

Abstract

The algebras Q_n describe the relationship between the roots and coefficients of a non-commutative polynomial. I. Gelfand, S. Gelfand, and V. Retakh have defined quotients of these algebras corresponding to graphs. In this work we find the Hilbert series of the class of algebras corresponding to the graph K_3 . We also show this algebra is Koszul.

1 Koszul Algebras

There are a number of equivalent definitions of Koszul algebras including this lattice definition from Ufnarovskij [7].

Definition 1. A quadratic algebra $A = \{V, R\}$ (where V is the span of the generators and R the span of the generating relations in $V \otimes V$) is Koszul if the collection of $n-1$ subspaces $\{V^{\otimes i-1} \otimes R \otimes V^{\otimes n-i-1}\}_i$ generates a distributive lattice in $V^{\otimes n}$ for any n .

In [4] the following criterion is given for distributivity of a modular lattice:

Theorem 1. Suppose $\{x_1, \dots, x_n\}$ generates the modular lattice Ω . If any proper subset of $\{x_1, \dots, x_n\}$ generates a distributive sublattice then Ω is distributive iff for any $2 \leq k \leq n-1$ the triple $x_1 \vee \dots \vee x_{k-1}, x_k, x_{k+1} \wedge \dots \wedge x_n$ is distributive.

Applying theorem 1 to definition 1 we get the following corollary we will use in various chapters throughout this work:

Corollary 1.1. The quadratic algebra $A = \{V, R\}$ (where V is the span of the generators and R the span of the generating relations in $V \otimes V$) is Koszul if $RV^{n-2} \cap VRV^{n-2} \cap \dots \cap V^{a-2}RV^{n-a}, V^{a-1}RV^{n-a-1}, V^aRV^{n-a-2} + \dots + V^{n-2}R$ is a distributive triple in V^n for any a and n with $2 \leq a \leq n-2$.

We will also need the following theorem from [7].

Theorem 2. A quadratic algebra A is Koszul iff its dual algebra A^* is Koszul. In the situation where they are both Koszul the Hilbert series of A is given by $\frac{1}{h(-x)}$ where $h(x)$ is the Hilbert series of A .

2 Q_n and $Q_n(G)$

Let $P(x) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - \dots + (-1)^n a_0$ be a polynomial over a division algebra. I. Gelfand and V. Retakh [2] studied relationships between the coefficients a_i and a generic set $\{x_1, \dots, x_n\}$ of solutions of $P(x) = 0$. For any ordering (i_1, \dots, i_n) of $\{1, \dots, n\}$ one can construct pseudoroots y_k , $k = 1, \dots, n$, (certain rational functions in x_{i_1}, \dots, x_{i_n}) that give a decomposition $P(t) = (t - y_n) \dots (t - y_2)(t - y_1)$ where t is a central variable.

In [3] I. Gelfand, V. Retakh, and R. Wilson introduced the algebra Q_n of all pseudo-roots of a generic noncommutative polynomial, determined a basis for this algebra and studied its structure. The algebras Q_n have a presentation given by generators $u(A)$, $\emptyset \neq A \subset [n]$ and relations

$$\sum_{C, D \subset A} [u(C \cup i), u(D \cup j)] = \left(\sum_{E \subset A} u(E \cup i \cup j) \right) \sum_{F \subset A} (u(F \cup i) - u(F \cup j))$$

for all $A \subset [n]$, $i, j \in [n] \setminus A$, $i \neq j$.

In [1] I. Gelfand, S. Gelfand, and V. Retakh introduced a class of quotient algebras of Q_n corresponding to graphs on n nodes. Let G be a graph with vertex set $[n] = \{1, 2, \dots, n\}$ and edge set E composed of elements of $P([n])$ with cardinality two (hence G has no loops of multiple edges). We can then consider the quotient algebra $Q_n(G)$ we get by adding the additional relations $u(\{i, j\}) = 0$ if $\{i, j\} \notin E$ to Q_n . The following theorem gives a nice presentation of the algebra $Q_n(G)$.

Theorem 3. [1] Let G be a graph on n nodes with edge set E . Then the algebra $Q_n(G)$ is generated by the elements $u(i)$ for $i \in [n]$ and $u(i, j)$ for $\{i, j\} \in E$ with the following relations (assume $u(i, j) = 0$ if $\{i, j\} \notin E$):

- (i) $[u(i), u(j)] = u(i, j)(u(i) - u(j))$ $i \neq j$, $i, j \in [n]$
- (ii) $[u(i, k), u(j, k)] + [u(i, k), u(j)] + [u(i), u(j, k)] = u(i, j)(u(i, k) - u(j, k))$ for distinct $i, j, k \in [n]$
- (iii) $[u(i, j), u(k, l)] = 0$ for distinct $i, j, k, l \in [n]$

3 The Algebra K_3

Here we consider the algebra that is generated by the graph K_3 . By theorem 3 this algebra has generators $u(1), u(2), u(3), u(12), u(13), u(23)$ together with the following relations in $V \otimes V$ which we will refer to as r_1 through r_5 .

$$\begin{aligned} r_1 &= [u(1), u(2)] + u(12)(u(2) - u(1)) = 0 \\ r_2 &= [u(2), u(3)] + u(23)(u(3) - u(2)) = 0 \\ r_3 &= [u(3), u(1)] + u(13)(u(1) - u(3)) = 0 \\ r_4 &= [u(12), u(23)] + [u(12), u(3)] + [u(1), u(23)] - u(13)(u(12) - u(23)) = 0 \\ r_5 &= [u(12), u(13)] + [u(12), u(3)] + [u(2), u(13)] - u(23)(u(12) - u(13)) = 0 \end{aligned}$$

We have only five relations because all other possible combinations in i) and ii) are linear combinations of these five. We also have no relations of type iii) because $n = 3$ and we do not have four distinct integers to work with.

We define an increasing filtration on K_3 by defining F_n to be the span of all monomials $u(A_1)u(A_2) \cdots u(A_k)$ such that $\sum_{i=1}^k |A_i| \leq n$. It is clear that our F_i are subspaces with the properties $\bigcup_i F_i = K_3$ and $F_i F_j \subseteq F_{i+j}$. We set the define F_0 to be the span of 1.

Now we form $gr(K_3)$ in the usual way. Take $G_i = F_i/F_{i-1}$ and set $gr(K_3) = \bigoplus_i G_i$ and then define multiplication in $gr(K_3)$ so for all $a \in F_i, b \in F_j, (a + F_{i-1})(b + F_{j-1}) = ab + F_{i+j-1}$. Note that there is a non-linear map $gr : K_3 \rightarrow gr(K_3)$ that sends $a \in F_i, a \notin F_{i-1}$ to $a + F_{i-1}$ in $gr(K_3)$ and sends 0 to 0.

4 A New Presentation of $gr(K_3)$

For ease of notation, let us temporarily set $a = u(1), b = u(2), c = u(3), d = u(12), e = u(23)$, and $f = u(13)$. Notice our relations in K_3 then become:

$$\begin{aligned} r_1 &= db - da + ab - ba \\ r_2 &= ec - eb + bc - cb \\ r_3 &= fa - fc + ca - ac \\ r_4 &= de - ed - fd + fe + dc - cd + ae - ea \\ r_5 &= df - fd - ed + ef + dc - cd + bf - fb \end{aligned}$$

Therefore in $gr(K_3)$ (if we allow each generator to represent itself under the image gr) we know the following relations hold:

$$\begin{aligned} d \cdot b &= da \\ e \cdot c &= eb \\ f \cdot c &= fa \\ f \cdot e &= fd - de + ed \\ f \cdot d &= df + ef - ed \end{aligned}$$

Now this list of relations might not be enough for a presentation of $gr(K_3)$, for other relations may hold true and be needed as well. Call the algebra generated by the five truncated relations above “chopped” K_3 (or $ch(K_3)$). We know that since $gr(K_3)$ is a quotient of $ch(K_3)$ (it has possibly more relations) we can verify that the two are equal by showing they have the same Hilbert series. Since $gr(K_3)$ has the same Hilbert series as K_3 we can compare the series for $ch(K_3)$ to the one for K_3 instead.

First let us use the diamond lemma to compute the Hilbert series of K_3 .

Theorem 4. *A basis for K_3 is given by the set of monomials in $T(V)$ containing none of the following substrings: cef, cd, cb, ca, bf, ba .*

Proof. Order the set of monomials first by monomial length, and then lexicographically with the ordering $c > b > e > f > a > d$ (which is actually $u(3) > u(2) > u(23) > u(13) > u(1) > u(12)$). We then have the following reductions (after replacing r_5 with $r_5 - r_4$):

$$\begin{aligned} cd &\rightarrow de - ed - fd + fe + dc + ae - ea \\ bf &\rightarrow fb + ae - ea + fe - ef + de - df \\ ba &\rightarrow ab + db - da \\ cb &\rightarrow bc + ec - eb \\ ca &\rightarrow ac + fc - fa \end{aligned}$$

This gives us cba and cbf as two ambiguities that need to be resolved. It is not hard to check that if we compute $c(ba) - (cb)a$ and reduce with the above five relations we get zero. However in order to resolve cbf it turns out we must add the relation: $cef = cfb + cfe + ace + fce - fae - cea + dce + de^2 + ae^2 + fe^2 - fde - dcf - def - aef + eaf - fef + fdf - bcf - ecf + efb + efe - e^2f - e^2a - edf$

However, adding this relation creates no new ambiguities. So we have found a basis in the set of monomials not containing the strings cef, cd, cb, ca, bf, ba . \square

Theorem 5. *The Hilbert series of K_3 is $H(x) = \frac{1}{x^3 - 6x^2 + 5x - 1}$.*

Proof. We must count the number of monomials of length n in $T(V)$ not containing any of the strings listed in theorem 4. Call such monomials the *valid* monomials of length n .

Let T_n be the number of valid monomials of length n .

Let J_n be the number of valid monomials of length n that begin with b .

Let K_n be the number of valid monomials of length n that begin with c .

We can note right away that $T_{n+1} = 4T_n + J_{n+1} + K_{n+1}$. Since words beginning with b can not be followed by a or f , we get $J_{n+1} = 2T_{n-1} + J_n + K_n$. Words beginning with c can be followed by c, f , or e , but in the e case they can not next be followed by f . This gives us $K_{n+1} = T_{n-1} + K_n + 3T_{n-2} + J_{n-1} + K_{n-1}$. By counting the valid words up to length three, we also get initial conditions. Thus we obtain the following system of recurrences:

$$\begin{aligned} T_{n+1} &= 4T_n + J_{n+1} + K_{n+1} \\ J_{n+1} &= 2T_{n-1} + J_n + K_n \\ K_{n+1} &= T_{n-1} + K_n + 3T_{n-2} + J_{n-1} + K_{n-1} \\ T_1 &= 6, T_2 = 31, T_3 = 157 \end{aligned}$$

To solve this system notice first that in the second equation $J_n + K_n = J_{n+1} - 2T_{n-1}$. Plugging this into our first equation gives $J_{n+2} = T_{n+1} - 2T_n$. We can use this to get rid of the J 's in the first and third equations to get the system:

$$\begin{aligned} T_{n+1} &= K_{n+1} + 5T_n - 2T_{n-1} \\ K_{n+1} &= K_n + K_{n-1} + T_{n-1} + 4T_{n-2} - 2T_{n-3} \end{aligned}$$

Solving the first for K_{n+1} and substituting into the second gives us $T_{n+1} = 6T_n - 5T_{n-1} + T_{n-2}$. Using generating functions and our initial conditions we can quickly find that the Hilbert series for K_3 is $H(x) = \frac{1}{x^3 - 6x^2 + 5x - 1}$. \square

Now we must find the Hilbert series of $ch(K_3)$. Consider an ordering first by monomial length and then lexicographically with $f > e > d > c > b > a$. We get the following reductions in $ch(K_3)$.

$$\begin{aligned} fe &\rightarrow ef + df - de \\ fd &\rightarrow df + ef - ed \\ db &\rightarrow da \\ ec &\rightarrow eb \\ fc &\rightarrow fa \end{aligned}$$

We have two ambiguities to resolve this time: fec and fdb . An attempt to resolve the fec ambiguity shows it necessary to add the relation $efb = efa - dfb + dfa$. With this new relation, we can resolve the fdb ambiguity. However, we created a new ambiguity by adding our efb relation. In order to resolve $fefb$ we must toss in the relation $effb = effa + dfda - dfdb + \frac{1}{2}edfb - \frac{1}{2}edfa - \frac{1}{2}ddfb + \frac{1}{2}ddfa$. We now have to worry about the ambiguity $feffb$. We can deal with all these ambiguities at once with the following lemma.

Lemma 1. *Suppose we need to resolve an ambiguity of the form $ef^nb = ev_n + dw_n - \alpha_n df^n b$ where v_n and w_n are linear combinations of monomials of length $n + 1$. Suppose also that the terms in v_n and w_n are all less than or equal then $f^n b$ and that α_n is a positive real number. Then the ambiguity $feffb$ can be resolved by adding a relation of the form $ef^{n+1}b = ev_{n+1} + dw_{n+1} - \alpha_{n+1} df^{n+1}b$ where v_{n+1} and w_{n+1} are linear combinations of monomials of length $n + 2$, the terms in v_{n+1} and w_{n+1} are all less than $f^{n+1}b$, and α_{n+1} is a positive real number.*

Proof. We have $(fe)f^nb - f(ef^nb) = ef^{n+1}b + df^{n+1}b - (d + f)(ev_n + dw_n - \alpha_n df^n b) = ef^{n+1}b + df^{n+1}b - dev_n - d^2w_n - \alpha_n d^2 f^n b - efv_n - dfv_n + dev_n - dfw_n - efw_n + \alpha_n df^{n+1}b + \alpha_n ef^{n+1}b - \alpha_n edf^n b$.

Thus $ef^{n+1}b + \alpha_n f^{n+1}b = d(-f^{n+1}b + dw_n + \alpha_n df^n b + fv_n + fw_n - \alpha_n f^{n+1}b) + e(fv_n + fw_n - dw_n + \alpha_n df^n b)$ and dividing by $1 + \alpha_n$ gives us:

$$ef^{n+1}b = \frac{1}{1+\alpha_n}e(fv_n + fw_n - dw_n + \alpha_n df^n b) + \frac{1}{1+\alpha_n}d(-f^{n+1}b + dw_n + \alpha_n df^n b + fv_n + fw_n) - \frac{\alpha_n}{1+\alpha_n}df^{n+1}b$$

Setting $v_n = \frac{1}{1+\alpha_n}(fv_n + fw_n - dw_n + \alpha_n df^n b)$, $w_n = \frac{1}{1+\alpha_n}(-f^{n+1}b + dw_n + \alpha_n df^n b + fv_n + fw_n)$ and taking α_{n+1} to the positive real constant $\frac{\alpha_n}{1+\alpha_n}$ completes the proof. \square

With this lemma, we see that the only bad words are ones containing strings of the following forms: fe, fd, db, ec, fc , or $ef^n b$ for $n \geq 1$. We can now use this to find the Hilbert series of $ch(K_3)$.

Proposition 4.1. *The Hilbert series of $ch(K_3)$ is equal to the Hilbert series of K_3 .*

Proof. We wish to count the strings not containing fe, fd, db, ec, fc , or $ef^n b$ for $n \geq 1$ as a substring. To count all such strings let T_n be the total number of valid monomials of length n . Let K_n be the number of valid monomials of length n beginning with d . Let L_n and M_n be the corresponding numbers for f and e respectively.

We can immediately see that $T_n = K_n + L_n + M_n + 3T_{n-1}$. We know that if a word begins with d , it can be followed by any smaller valid word not beginning with b . This gives us $K_n = 2T_{n-2} + K_{n-1} + L_{n-1} + M_{n-1}$. The words beginning with f can be followed by f, a or b giving us $L_n = 2T_{n-2} + L_{n-1}$.

Finally we must count the words beginning with e . If e is followed by any valid word not beginning with f or c then we are okay. There will be $2T_{n-2} + K_{n-1} + M_{n-1}$ of these. If the second letter does happen to be f then the next letter can only be a or f . If it is a then we can follow up with any valid word (which adds T_{n-3} to the equation), but if it is f we are once again in an a or f situation. This time the a case ends up adding T_{n-4} . We can repeat this down the line to add $T_{n-5} + T_{n-6} + \dots + T_1$ and finally we add 2 (or $2T_0$) for the $ef \dots fa$, and $ef \dots ff$ cases. This gives us $M_n = 2T_{n-2} + K_{n-1} + M_{n-1} + \sum_{k=3}^n T_{n-k} + T_0$.

We must now solve the following system of recurrences:

$$T_n = K_n + L_n + M_n + 3T_{n-1}$$

$$K_n = 2T_{n-2} + K_{n-1} + L_{n-1} + M_{n-1}$$

$$L_n = 2T_{n-2} + L_{n-1}$$

$$M_n = 2T_{n-2} + K_{n-1} + M_{n-1} + \sum_{k=3}^n T_{n-k} + T_0$$

To do this, first we must get rid of the summation for M_n . Set $R_n = M_n - M_{n-1}$ which equals $2T_{n-2} + K_{n-1} + M_{n-1} + \sum_{k=3}^n T_{n-k} + T_0 - 2T_{n-3} - K_{n-2} - M_{n-2} - \sum_{k=3}^{n-1} T_{n-1-k} - T_0 = 2T_{n-2} - 2T_{n-3} + K_{n-1} - K_{n-2} + M_{n-1} - M_{n-2} + \sum_{k=3}^n T_{n-k} - \sum_{k=4}^n T_{n-k} = 2T_{n-2} - 2T_{n-3} + K_{n-1} - K_{n-2} + R_{n-1} + T_{n-3}$ so $R_n = 2T_{n-2} - 2T_{n-3} + K_{n-1} - K_{n-2} + R_{n-1}$.

Now we can get rid of our M_n terms altogether by replacing T_n and K_n with $T_n - K_n = 3T_{n-1} - 2T_{n-2} + K_n - K_{n-1} + L_n - L_{n-1} + R_n$ and $T_n - K_{n+1} = 3T_{n-1} + K_n + L_n + M_n - 2T_{n-1} - K_n - L_n - M_n = T_{n-1}$.

By plugging $L_n - L_{n-1} = 2T_{n-2}$ into our T_n equation we are left with the following system:

$$T_n = 3T_{n-1} + 2K_n - K_{n-1} + R_n$$

$$R_n = 2T_{n-2} - T_{n-3} + R_{n-1} + K_{n-1} - K_{n-2}$$

$$K_n = T_{n-1} - T_{n-2}$$

Substituting for K_n leads to the system:

$$T_n = 5T_{n-1} - 3T_{n-2} + T_{n-3} + R_n$$

$$R_n = 3T_{n-2} - 3T_{n-3} + T_{n-4} + R_{n-1}$$

We can now solve for R_n in the first equation and substitute into the second equation to get $T_n = 6T_{n-1} - 5T_{n-2} + T_{n-3}$. This is the same recurrence we used to generate the Hilbert series of K_3 . Checking the length one, two, and three cases gives the same initial conditions as well. Therefore the two algebras must have the same Hilbert series. □

Corollary 4.1. $ch(K_3) \cong gr(K_3)$.

Proof. We know $ch(K_3)$ has the same graded dimension as K_3 which has the same graded dimension as $gr(K_3)$. Since $ch(K_3)$ is a quotient of $gr(K_3)$ with the same graded dimension, the two must be isomorphic. □

From now on we will list the chopped relations for a presentation of $gr(K_3)$.

5 Reduction to $gr(K_3)$

The following corollary from [5] will be useful here.

Corollary 5.1. *Let $0 = F_0W \subset F_1W \subset \dots \subset F_{l-1}W \subset F_lW$ be a filtered vector space and X_1, \dots, X_n be a collection of subspaces; then the following two conditions are equivalent:*

(a) *the whole set of subspaces $F_0W, \dots, F_lW, X_1, \dots, X_n \subset W$ is distributive.*

(b) *the associated graded collection grX_1, \dots, grX_n in the associated graded vector space grW is distributive are for any $1 \leq i < j \leq n$ either of the two equivalent conditions holds:*

$$gr(X_i + X_j) = grX_i + grX_j \text{ or } gr(X_i \cap X_j) = grX_i \cap grX_j.$$

Hence, if the set $\{grX_i\}_i$ generates a distributive lattice in grV , and $gr(X_i \cap X_j) = gr(X_i) \cap gr(X_j)$ for all i and j , then the set $\{X_i\}_i$ generates a distributive lattice in V . We wish to check that $gr(RV \cap VR) = gr(RV) \cap gr(VR)$. By looking at gr as a function and only using the basic rules for set maps and intersections we see that $gr(RV \cap VR) \subseteq gr(RV) \cap gr(VR)$. It will be enough for us to show that the dimension of $gr(RV) \cap gr(VR)$ is one and that $gr(RV \cap VR)$ is not the zero subspace.

For the second part, notice that the vector $r_1c + r_2a + r_3b + r_4(c-b) + r_5(a-c)$ is equal to the vector $ar_2 + br_3 + cr_1 + d(r_2 + r_3) + e(r_1 + r_3) + f(r_1 + r_2)$ where r_1, \dots, r_5 are the relations we defined earlier for K_3 . This shows that $RV \cap VR$ is not zero, and thus $gr(RV \cap VR)$ is not the zero subspace.

Proposition 5.1. $\dim gr(RV) \cap gr(VR) = 1$

Proof. In the last section we showed that $gr(R)$ is the span of $\{fe - ef - df + de, fd - df - ef + ed, d(b-a), e(c-b), f(a-c)\}$. Since these are the relations we will be working with throughout the rest of this paper, we officially set:

$$\begin{aligned} r_1 &= db - da \\ r_2 &= ec - eb \\ r_3 &= fa - fc \\ r_4 &= de - ed - fd + fe \\ r_5 &= df - fd - ed + ef \end{aligned}$$

Call the span of these five relations S (so $S = gr(R)$). Our goal is to show $SV \cap VS$ is of dimension one.

Suppose $x \in SV \cap VS$ and $x \neq 0$. Since all the monomials in VS contain no a, b , or c in the middle spot, we can replace SV with $sp\{r_4, r_5\} \otimes V$. As all the monomials in SV contain no a, b , or c in the first slot we can replace VS with $sp\{d, e, f\} \otimes S$.

Suppose $x \in sp\{r_4, r_5\} \otimes V \cap sp\{d, e, f\} \otimes S$. Then we can write $x = r_4v_1 + r_5v_2$ for some $v_1, v_2 \in V$. Now if the coefficient of c in v_1 was nonzero we would have an fee term with nothing else that could cancel it out. Hence we would have an fee appearing in VS which is not possible. If the coefficient of f in v_1 was nonzero then we would have a def which could only happen in VS if the coefficient of dr_5 was nonzero. But this would give us a nonzero dee term with nothing to cancel it out, which is also not possible. If the coefficient of d was nonzero we would have an fed appearing which implies that $\alpha fr_4 + \beta fr_5$ appears in VS with $\alpha \neq -\beta$. As ffe can not appear, α must be zero, so $\beta \neq 0$. But then we would have an $-\beta ffd$ appearing with nothing to cancel it out with. This shows that $v_1 \in sp\{a, b, c\}$. A similar argument shows that v_2 is in this same span. We know now that $x \in SV \cap VS$ implies $x \in sp\{r_4, r_5\} \otimes sp\{a, b, c\} \cap sp\{d, e, f\} \otimes S$. We can replace this last S with $sp\{r_1, r_2, r_3\}$ after noticing that only a, b , and c can now appear in the last slot. So far we know $SV \cap VS = sp\{r_4, r_5\} \otimes sp\{a, b, c\} \cap sp\{d, e, f\} \otimes sp\{r_1, r_2, r_3\}$.

Now suppose $x = r_4v_1 + r_5v_2$ where $v_1, v_2 \in sp\{a, b, c\}$ and $x \neq 0$. Suppose also that the coefficient of b in v_1 is 0. Then no deb or feb can appear in x (with a non-zero coefficient). Looking in VS we see this means dr_2 and fr_2 must have coefficients of zero. This means no dec or fec can appear in x either. Hence v_1 is a constant multiple of a . However this constant must be 0, otherwise the term dea would appear, and there is no way to achieve that in VS . So in this case v_1 is 0 and $x = r_5v_2$. Then v_2 would have to be a multiple of a since no dfb and fdc can occur. So x is a constant multiple of r_5a . But r_5a is not in VS so we have reached a contradiction.

We now know that if $x = r_4v_1 + r_5v_2 \in SV \cap VS$ then the coefficient of b in v_1 is non-zero, so we can scale x to make this coefficient 1. Hence $deb + feb$ appears in x . Looking in SV we see this implies $-dec - fec$ appears as well. This implies the coefficient of c in v_1 is -1 . As fdc can not appear, this term must cancel, meaning the coefficient of c in v_2 is 1. This means $dfc + efc$ appears in x . Looking in SV we see $-dfa - efa$ appears, implying the coefficient of a in v_2 is -1 . Since the coefficients of a in v_1 and b in v_2 must be 0 (look at SV to see this) we get that $x = r_4(b-c) + r_5(c-a)$. It is easy to see this x is in VS because it is equal to $(e+f)r_1 - (d+f)r_2 - (d+e)r_3$. □

6 A Reduction to Two Cases

Recall from corollary 1.1 that K_3 will be Koszul if we show that $RV^{n-2} \cap VRV^{n-2} \cap \dots \cap V^{a-2}RV^{n-a}, V^{a-1}RV^{n-a-1}, V^aRV^{n-a-2} + \dots + V^{n-2}R$ is a distributive triple in V^n for any n and $2 \leq a \leq n-2$.

We will require the following lemma of Serconek and Wilson (lemma 1.1 from [6]):

Lemma 2. *If*

1) $V = \sum_{i \in I} V_i$ *is graded as a vector space*

2) X_j *is a collection of subspaces of* V

3) *Each* $X_j = \sum_{i \in I} (X_j \cap V_i)$

then $\{X_j\}_j$ *is distributive if and only if for all* $i \in I$, $\{X_j \cap V_i\}_j$ *is distributive in* V_i .

We now choose a particular $\{1, 2\}^n$ grading of V to apply this lemma to. Set $V_{(i_1, i_2, \dots, i_n)}^n$ to be the span of all monomials $u(A_1)u(A_2) \cdots u(A_n)$ so that $|A_k| = i_k$. For example, in the $n = 2$ case r_1, r_2 , and r_3 are in the $V_{(2,1)}^2$ space and r_4 and r_5 are in the $V_{(2,2)}^2$ space. Hence $R = (R \cap V_{(2,1)}^2) + (R \cap V_{(2,2)}^2)$. This shows that property three of lemma 2 will apply to our sets $\{V^{a-2}RV^{n-a}\}$.

We have to show that $\{RV^{n-2} \cap (V^n)_\alpha, \dots, V^{n-2}R \cap (V^n)_\alpha\}$ generates a distributive lattice in $(V^n)_\alpha$ for $\alpha \in \{1, 2\}^n$. Notice that we need only check the $\alpha = (i_1, i_2, \dots, i_n)$ such that $i_1 \geq i_2 \geq \dots \geq i_n$. This is because $R \in V_{(2,1)}^2 + V_{(2,2)}^2$ so if a $(1, 2)$ appears somewhere in the string α then one of our $V^{a-2}RV^{n-a} \cap (V^n)_\alpha$ will be zero. Since we can show proper subsets of $\{RV^{n-2} \cap (V^n)_\alpha, \dots, V^{n-2}R \cap (V^n)_\alpha\}$ are distributive, we are done for such α .

Next notice that the subspace $RV \cap VR$ we found earlier is contained in $V_{(2,2,1)}^3$. Combining corollary 1.1 with lemma 2, we have to check that $RV^{n-2} \cap \dots \cap V^{a-2}RV^{n-a} \cap (V^n)_\alpha, V^{a-1}RV^{n-a-1} \cap (V^n)_\alpha, (V^aRV^{n-a-2} \cap (V^n)_\alpha) + \dots + (V^{n-2}R \cap (V^n)_\alpha)$ is a distributive triple for any decreasing α and $2 \leq a \leq n-2$. However if the last two digits of α are $(1, 1)$ then the last term of the third element $(V^{n-2}R \cap (V^n)_\alpha) = 0$ and we will be done because proper subsets are distributive. We are done with all cases except when α contains a two in the second to last spot. This leaves $\alpha = (2, 2, \dots, 2, 1)$ and $\alpha = (2, 2, \dots, 2)$.

Next suppose $a > 2$. Notice that our first term in our triple is $RV^{n-2} \cap VRV^{n-3} \cap \dots \cap V^{a-2}RV^{n-a} \cap (V^n)_\alpha$ which is contained in $(RV \cap VR)V^{n-3} \cap (V^n)_\alpha$. Since α can not start out with $(2, 2, 1)$ and $RV \cap VR$ is contained in this graded space, this term must be zero. Hence we need only check the case where $a = 2$.

From here on when we say a set $\{X_1, \dots, X_n\}$ is distributive in the α case we mean that the set $\{X_1 \cap (V^n)_\alpha, \dots, X_n \cap (V^n)_\alpha\}$ is distributive. Thus we must show $RV^{n-2}, VRV^{n-3}, V^2RV^{n-4} + \dots + V^{n-2}R$ is a distributive triple in the $(2, 2, \dots, 2)$ and $(2, 2, \dots, 2, 1)$ cases. This amounts to showing $RV^{n-2} \cap (VRV^{n-3} + V^2RV^{n-4} + \dots + V^{n-2}R) = (RV^{n-2} \cap VRV^{n-3}) + (RV^{n-2} \cap (V^2RV^{n-4} + \dots + V^{n-2}R))$ for those two cases. But as $RV \cap VR$ is in $V_{(2,2,1)}^3$, we know that $RV^{n-2} \cap VRV^{n-3} = 0$ in these cases. This simplifies what we must show to $RV^{n-2} \cap (VRV^{n-3} + V^2RV^{n-4} + \dots + V^{n-2}R) = (RV^{n-2} \cap (V^2RV^{n-4} + \dots + V^{n-2}R))$. And since the right side is contained in the left we only have one direction left to show. We need show that if $x \in RV^{n-2}$ and $x \in VRV^{n-3} + \dots + V^{n-2}$ then $x \in V^2RV^{n-4} + \dots + V^{n-2}R$.

Introducing a little more terminology makes this statement simpler. Define $W_k = RV^{k-2} + VRV^{k-3} + \dots + V^{k-2}R$ for $k \geq 2$ and set W_k to the zero subspace otherwise. Then we must show that $X \in RV^{n-2} \cap VW_{n-1} \implies x \in V^2W_{n-2}$ for our two cases. Notice also that this statement is trivial for $n < 4$.

Next we look for a more natural spanning set for R . Let σ be the permutation $(1, 2, 3)$. Notice that the map T sending $u(A)$ to $u(\sigma(A))$ sends $sp\{r_1, r_2, r_3\}$ and $sp\{r_4, r_5\}$ both back to themselves. Hence R is T invariant. Letting ω be a primitive cube root of one, the following elements form a basis for R consisting only of eigenvectors. Set:

$$\begin{aligned} u_1 &= u(12) + u(23) + u(13) \\ u_\omega &= u(12) + \omega u(23) + \omega^2 u(13) \\ u_{\omega^2} &= u(12) + \omega^2 u(23) + \omega u(13) \\ v_1 &= u(1) + u(2) + u(3) \\ v_2 &= u(1) + \omega u(2) + \omega^2 u(3) \\ v_3 &= u(1) + \omega^2 u(2) + \omega u(3) \end{aligned}$$

then our relations become

$$\begin{aligned} r_1 &= (u_1 + u_\omega + u_{\omega^2})(\omega - 1)v_1 + (\omega^2 - 1)v_2 \\ r_2 &= (\omega^2 u_1 + u_\omega + \omega u_{\omega^2})(\omega^2 - \omega)v_1 + (\omega - \omega^2)v_2 \\ r_3 &= (\omega u_1 + u_\omega + \omega^2 u_{\omega^2})((1 - \omega^2)v_1 + (1 - \omega)v_2) \\ r_4 &= u_\omega^2 - 2u_1 u_\omega + u_\omega u_1 \\ r_5 &= u_\omega^2 - 2u_1 u_{\omega^2} + u_\omega u_1 \end{aligned}$$

If we set $a = v_1, b = v_2, d = u_1, e = u_\omega, f = u_{\omega^2}$ then we get the simpler looking set:

$$\begin{aligned} r_1 &= (e + f + d)((\omega - 1)a + (\omega^2 - 1)b) \\ r_2 &= (e + \omega f + \omega^2 d)((\omega^2 - \omega)a + (\omega - \omega^2)b) \\ r_3 &= (e + \omega^2 f + \omega d)((1 - \omega^2)a + (1 - \omega)b) \\ r_4 &= f^2 - 2de - ed \\ r_5 &= e^2 - 2df + fd \end{aligned}$$

Keep in mind that r_4 and r_5 still sit in $V_{(2,2)}^2$ and r_1, r_2 , and r_3 still sit in $V_{(2,1)}^2$ since d, e , and f are multiples of $u(12)$, $u(23)$, and $u(12)$ and a, b , and c are multiples of $u(1)$, $u(2)$, and $u(3)$. This will be our spanning set for R as we move on to the two last cases.

7 The Two Last Cases

Our main goal is to prove the following lemma in the $(2, 2, \dots, 2)$ and $(2, 2, \dots, 1)$ cases.

Lemma 3. *Suppose $n \geq 2$ then*

- a) If $z_1, z_2 \in V^n, dz_1 + ez_2 \in W_{n+1}$ then $z_1, z_2 \in W_n$*
- b) If $y_1, y_2 \in V^n, ey_1 + fy_2 \in W_{n+1}$ then $y_1, y_2 \in W_n$*
- c) If $x_1, x_2 \in V^n, dx_1 + fx_2 \in W_{n+1}$ then $x_1, x_2 \in W_n$*

Proof. Suppose we knew the lemma was true for either $n = 1$ or $n = 2$. Then we can assume by induction that the lemma holds for $n - 1$. Suppose we are in the a) case and set $z = dz_1 + ez_2 \in W_{n+1}$. Then we can write $z = r_4h_1 + r_5h_2 + VW_n = f(fh_1 + dh_2) + e(eh_2 + dh_1) - 2d(eh_1 + fh_2) + VW_n$. This means $fh_1 + dh_2 \in W_n$. Since $z \in dV^n + eV^n$ we know that $f(fh_1 + dh_2) \in VW_n$ so $fh_1 + dh_2 \in W_n$. By our inductive hypothesis $h_1, h_2 \in W_{n-1}$. So $z \in r_4W_{n-1} + r_5W_{n-1} + VW_n \subset VW_n$ and hence $z_1, z_2 \in W_n$. The b) and c) cases are similar.

We have left to find basis cases for lemma. In the $(2, 2, \dots, 2)$ situation we can find one when $n = 1$. Assume $dz_1 + ez_2 \in W_2 = R = \alpha r_4 + \beta r_5$. If $\alpha \neq 0$ we would have an f^2 appearing which we could not cancel, and hence a contradiction. Similarly β must be 0 as well and hence z_1 and z_2 are both in $W_1 = 0$. \square

Now that we know that the lemma is true in both cases we can prove the following proposition thus completing our proof that K_3 is Koszul.

Proposition 7.1. *If $n \geq 4, x \in RV^{n-2} \cap V_{n-1}$ then $x \in V^2W_{n-2}$*

Proof. Since $x \in RV^{n-2}$ we can write $x = r_4h_1 + r_5h_2 = (f^2 - 2de - ed)h_1 + (e^2 - 2df + fd)h_2 = f(fh_1 + dh_2) + e(eh_2 + dh_1) - 2d(eh_1 + fh_2)$. Since $x \in VW_{n-1}$ we know that $fh_1 + dh_2, eh_2 + dh_1$, and $eh_1 + fh_2$ are all in W_{n-1} . Since $n - 1 \geq 3$, lemma 3 applies and thus $h_1, h_2 \in W_{n-2}$. Thus $x \in r_4W_{n-2} + r_5W_{n-2} \subset V^2W_{n-2}$ and we are done. \square

Theorem 6. *K_3 is Koszul.*

References

- [1] Israel Gelfand, Sergei Gelfand, and Vladimir Retakh. Noncommutative algebras associated to complexes and graphs. *Selecta Math. (N.S.)*, 7(4):525–531, 2001.
- [2] Israel Gelfand and Vladimir Retakh. Noncommutative Vieta theorem and symmetric functions. In *The Gelfand Mathematical Seminars, 1993–1995*, Gelfand Math. Sem., pages 93–100. Birkhäuser Boston, Boston, MA, 1996.
- [3] Israel Gelfand, Vladimir Retakh, and Robert Lee Wilson. Quadratic linear algebras associated with factorizations of noncommutative polynomials and noncommutative differential polynomials. *Selecta Math. (N.S.)*, 7(4):493–523, 2001.
- [4] Romolo Musti and Ettore Buttafuoco. Sui subreticoli distributivi dei reticoli modulari. *Boll. Un. Mat. Ital. (3)*, 11:584–587, 1956.
- [5] Positselski L. Polishchuk A. Quadratic algebras. *Preprint*, 1996.
- [6] Shirlei Serconek and Robert Lee Wilson. The quadratic algebras associated with pseudo-roots of noncommutative polynomials are Koszul algebras. *J. Algebra*, 278(2):473–493, 2004.
- [7] V. A. Ufnarovskij. Combinatorial and asymptotic methods in algebra [MR1060321 (92h:16024)]. In *Algebra, VI*, volume 57 of *Encyclopaedia Math. Sci.*, pages 1–196. Springer, Berlin, 1995.